Explicit analytical solutions fof transport equations considering non-Fourier and non-Fick effects in porous media*

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Abstract Based on the method of separation variables with addition developed in recent years, new methods of separation variables are proposed, and two algebraically explicit analytical solutions to the general partial differential equation set of non-Fourier and non-Fick heat and mass transfer in porous media drying are derived. The physical meaning of these solutions is simple and clear, and they are valuable for computational heat and mass transfer as benchmark solutions.

Keywords: non-Fourier effects, non-Fick effects, porous media, heat and mass transfer, analytical solution.

Drying process in porous media widely exists in many fields such as energy, chemistry, biology, medicine and architectonics. Researches into this kind of phenomenon are traditionally based on Fourier law and Fick law. These two laws include the hypotheses of infinite propagation of heat and mass disturbance, respectively. Such an approximate approach is precise enough for the conventional unsteady heat and mass transfer. However, for high intensive and rapid drying process, it could lead to errors to some extent. Therefore, it is necessary to consider the non-linear relationship between the propagation of heat and mass flux and their gradients (the relaxation effects).

The general equation set which describes non-Fourier and non-Fick heat and mass transfer for capillary porous media has been given in Ref. [1]. To simplify the solution of the equation set, Ref. [1] gave two assumptions: in a high intensive and rapid drying process, the temperature gradient has much smaller effect on gas transfer process compared with the pressure gradient; the combined substances in the capillary porous media exist mainly in liquid form, and the content of initial water steam is very small and can be neglected. Based on these two assumptions, Ref. [1] proposed an equation set of one-dimensional unsteady temperature field and moisture field:

$$\frac{\partial\theta}{\partial t}+\tau_q\frac{\partial^2\theta}{\partial t^2}=a_{11}\frac{\partial^2\theta}{\partial x^2}+a_{12}\frac{\partial^2 u}{\partial x^2}-a_{13}\frac{\partial u}{\partial t},$$

$$\frac{\partial u}{\partial t}+\tau_m\frac{\partial^2 u}{\partial t^2}=a_{21}\frac{\partial^2\theta}{\partial x^2}+a_{22}\frac{\partial^2 u}{\partial x^2},\qquad(1)$$

where

$$a_{11}=a+\frac{\lambda_{qg}\varphi a_m\delta}{C_q K_g(1-\varphi)};$$

$$a_{12}=\xi a+\frac{\lambda_{qg}\varphi a_m}{C_q K_g(1-\varphi)};$$

$$a_{13}=\frac{r\varphi}{C_q};\quad a_{21}=\frac{a_m\delta}{1-\varphi};\quad a_{22}=\frac{a_m}{1-\varphi}.$$

Here, θ, u, t, x are temperature, moisture, time and geometric coordinates, respectively, τ_q is the heat relaxation time and τ_m is the mass relaxation time; a is the coefficient of temperature conductivity, ξ is the moisture gradient coefficient, C_q is the reduced specific heat of infinitesimal volume, φ is the phase transformation coefficient, r is the latent heat of vaporization, a_m is the mass diffusivity, δ is the thermal gradient coefficient, R is the gas constant, λ_{qg} is the coefficient of heat conductivity of gaseous phase mass, and K_g is the coefficient of infiltration flow.

Many analytical solutions play a key role in the early development of fluid mechanics as well as of the heat conduction[2,3]. Besides their irreplaceable theoretical meaning, analytical solutions can also be applied as benchmark solutions to develop numerical methods. Therefore, possible algebraically explicit analytical solutions for equation set (1) will be de-

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duced in this paper to promote the researches in heat and mass transfer of drying process.

Ref. [4] reports two algebraically explicit analytical solutions of simplified form of equation set (1) in which $\tau_q = \tau_m = a_{13} = 0$, which means non-Fourier and non-Fick effects are not considered. However, no analytical solution has been reported for equation set (1). In this paper, two sets of algebraically explicit analytical solutions are deduced for equation set (1) and their physical meaning is discussed. These analytical solutions are valuable to deepen the understanding of non-Fourier and non-Fick heat and mass transfer in the rapid drying process of porous media. They can also be applied to check the accuracy, convergence and stability of relative numerical computation solutions. Therefore, these analytical solutions have high reference value. For a given analytical solution, its correctness can be proved easily by substituting it into the equation set.

1 The first set of analytical solution

For an unknown function $f(x, y)$ in partial differential equations, the common method of separation variables is to assume $f = X(x) \circ Y(y)$. Another approach is to assume $f = X(x) + Y(y)$, which is named the method of separation variables with addition[5]. Because the assumptions of function forms are measures to separate variables, we can, in order to realize separation, assume other function forms according to the modality of equations. It has been proved that the method of separation variables with addition can apply to various sorts of equations[6—13]. Based on the method of separation variables with addition, and according to the particularity of the equation set (1), assuming:

$$\theta = T_1(t) + X_1(x) + (K_1 x^2 + K_2 x)t,$$
$$u = T_2(t) + X_2(x) + (K_3 x^2 + K_4 x)t,$$

the variables of equation set (1) can be separated as follows:

$$T_1' + \tau_q T_1'' - 2a_{11}K_1 t - 2a_{12}K_3 t + a_{13} T_2' = K_5 = a_{11}X_1'' + a_{12}X_2'' - a_{13}(K_3 x^2 + K_4 x) - (K_1 x^2 + K_2 x), \quad (2)$$

$$T_2' + \tau_m T_2'' - 2a_{21}K_1 t - 2a_{22}K_3 t = K_6 = a_{21}X_1'' + a_{22}X_2'' - (K_3 x^2 + K_4 x), \quad (3)$$

where K_1—K_6 are arbitrary constants. In the following equations, K_i represent different arbitrary constants.

From the left side of Eq. (3), we have

$$T_2 = (K_1 a_{21} + K_3 a_{22})t^2 + (K_6 - 2K_1 a_{21}\tau_m - 2K_3 a_{22}\tau_m)t + K_7 \exp\left(-\frac{t}{\tau_m}\right) + K_{12}. \quad (4)$$

Substituting Eq. (4) into the left side of Eq. (2), the following result can be obtained:

$$T_1 = (K_1 a_{11} + K_3 a_{12} - K_1 a_{13} a_{21} - K_3 a_{13} a_{22})t^2 + (K_5 - K_6 a_{13} + 2K_1 a_{13} a_{21}\tau_m + 2K_3 a_{13} a_{22}\tau_m - 2K_1 a_{11}\tau_q - 2K_3 a_{12}\tau_q + 2K_1 a_{13} a_{21}\tau_q + 2K_3 a_{13} a_{22}\tau_q)t - \frac{a_{13}K_7\tau_m}{\tau_m - \tau_q}\exp\left(-\frac{t}{\tau_m}\right) + K_8 \exp\left(-\frac{t}{\tau_q}\right) + K_{11}. \quad (5)$$

From the right side of Eqs. (2) and (3), we have

$$X_1 = [(K_3 a_{12} - K_3 a_{22} a_{13} - K_1 a_{22})x^4 + 2(K_4 a_{12} - K_4 a_{13} a_{22} - K_2 a_{22})x^3 + 6(K_6 a_{12} - K_5 a_{22})x^2 + K_9 x] / 12(a_{12}a_{21} - a_{11}a_{22}), \quad (6)$$

$$X_2 = [(K_3 a_{21} a_{13} + K_1 a_{21} - K_3 a_{11})x^4 + 2(K_2 a_{21} + K_4 a_{13} a_{21} - K_4 a_{11})x^3 + 6(K_5 a_{21} - K_6 a_{11})x^2 + K_{10} x] / 12(a_{12}a_{21} - a_{11}a_{22}). \quad (7)$$

Combining Eqs. (4)—(7), a possible algebraically explicit analytical solution of equation set (1) can be obtained:

$$\theta = (K_1 a_{11} + K_3 a_{12} - K_1 a_{13} a_{21} - K_3 a_{13} a_{22})t^2 + (K_5 - K_6 a_{13} + 2K_1 a_{13} a_{21}\tau_m + 2K_3 a_{13} a_{22}\tau_m - 2K_1 a_{11}\tau_q - 2K_3 a_{12}\tau_q + 2K_1 a_{13} a_{21}\tau_q + 2K_3 a_{13} a_{22}\tau_q)t - \frac{a_{13}K_7\tau_m}{\tau_m - \tau_q}\exp\left(-\frac{t}{\tau_m}\right) + K_8 \exp\left(-\frac{t}{\tau_q}\right) + (K_1 x^2 + K_2 x)t + [(K_3 a_{12} - K_3 a_{22} a_{13} - K_1 a_{22})x^4 + 2(K_4 a_{12} - K_4 a_{13} a_{22} - K_2 a_{22})x^3 + 6(K_6 a_{12} - K_5 a_{22})x^2 + K_9 x] / 12(a_{12}a_{21} - a_{11}a_{22}) + K_{11},$$

$$u = (K_1 a_{21} + K_3 a_{22})t^2 + (K_6 - 2K_1 a_{21}\tau_m - 2K_3 a_{22}\tau_m)t + K_7 \exp\left(-\frac{t}{\tau_m}\right) + (K_3 x^2 + K_4 x)t + [(K_3 a_{21} a_{13} + K_1 a_{21} - K_3 a_{11})x^4 + 2(K_2 a_{21} + K_4 a_{13} a_{21} - K_4 a_{11})x^3$$

$$+6(K_5a_{21}-K_6a_{11})x^2+K_{10}x]$$
$$/12(a_{12}a_{21}-a_{11}a_{22})+K_{12}. \quad (8)$$

Solution (8) includes terms whose denominators are $(a_{12}a_{21}-a_{11}a_{22})$. Thereby it is necessary that $(a_{12}a_{21}-a_{11}a_{22})\neq 0$. According to Ref. [4], in general $(a_{12}a_{21}-a_{11}a_{22})<0$, so this solution is effective.

The initial condition for solution (8) is:

$$t=0,$$
$$\theta=[(K_3a_{12}-K_3a_{22}a_{13}-K_1a_{22})x^4+2(K_4a_{12}-K_4a_{13}a_{22}-K_2a_{22})x^3+6(K_6a_{12}-K_5a_{22})x^2+K_9x]/12(a_{12}a_{21}-a_{11}a_{22})+K_{11}-\frac{a_{13}K_7\tau_m}{\tau_m-\tau_q}+K_8,$$
$$u=[(K_3a_{21}a_{13}+K_1a_{21}-K_3a_{11})x^4+2(K_2a_{21}+K_4a_{13}a_{21}-K_4a_{11})x^3+6(K_5a_{21}-K_6a_{11})x^2+K_{10}x]/12(a_{12}a_{21}-a_{11}a_{22})+K_{12}+K_7.$$

It means that the temperature field and moisture field are initially uneven commonly, but this distribution is possible in physics.

And the boundary condition for solution (8) is:

$$x=0,$$
$$\theta=(K_1a_{11}+K_3a_{12}-K_1a_{13}a_{21}-K_3a_{13}a_{22})t^2+(K_5-K_6a_{13}+2K_1a_{13}a_{21}\tau_m+2K_3a_{13}a_{22}\tau_m-2K_1a_{11}\tau_q-2K_3a_{12}\tau_q+2K_1a_{13}a_{21}\tau_q+2K_3a_{13}a_{22}\tau_q)t-\frac{a_{13}K_7\tau_m}{\tau_m-\tau_q}\exp\left(-\frac{t}{\tau_m}\right)+K_8\exp\left(-\frac{t}{\tau_q}\right)+K_{11},$$
$$u=(K_1a_{21}+K_3a_{22})t^2+(K_6-2K_1a_{21}\tau_m-2K_3a_{22}\tau_m)t+K_7\exp\left(-\frac{t}{\tau_m}\right)+K_{12},$$
$$x=1,$$
$$\theta=(K_1a_{11}+K_3a_{12}-K_1a_{13}a_{21}-K_3a_{13}a_{22})t^2+(K_5-K_6a_{13}+2K_1a_{13}a_{21}\tau_m+2K_3a_{13}a_{22}\tau_m-2K_1a_{11}\tau_q-2K_3a_{12}\tau_q+2K_1a_{13}a_{21}\tau_q+2K_3a_{13}a_{22}\tau_q+K_1+K_2)t-\frac{a_{13}K_7\tau_m}{\tau_m-\tau_q}\exp\left(-\frac{t}{\tau_m}\right)+K_8\exp\left(-\frac{t}{\tau_q}\right)+[K_3a_{12}-K_3a_{22}a_{13}-K_1a_{22}+2(K_4a_{12}-K_4a_{13}a_{22}-K_2a_{22})+6(K_6a_{12}-K_5a_{22})+K_9]/12(a_{12}a_{21}-a_{11}a_{22})+K_{11},$$
$$u=(K_1a_{21}+K_3a_{22})t^2+(K_6-2K_1a_{21}\tau_m-2K_3a_{22}\tau_m+K_3+K_4)t+K_7\exp\left(-\frac{t}{\tau_m}\right)+[(K_3a_{21}a_{13}+K_1a_{21}-K_3a_{11})+2(K_2a_{21}+K_4a_{13}a_{21}-K_4a_{11})+6(K_5a_{21}-K_6a_{11})+K_{10}]/12(a_{12}a_{21}-a_{11}a_{22})+K_{12}.$$

It means that the temperature and moisture at $x=0$ and $x=1$ are transient in general, but this distribution is also possible in physics.

Assuming some constants have the following relations between them: $K_3(a_{12}-a_{22}a_{13})=K_1a_{22}$, $K_4(a_{12}-a_{13}a_{22})=K_2a_{22}$, $K_6a_{12}=K_5a_{22}$ and $K_9=0$, solution (8) can be simplified as:

$$\theta=K_1(a_{11}+a_{22}-a_{13}a_{21})t^2+\left[K_5\left(1-\frac{a_{13}a_{22}}{a_{12}}\right)+2K_1\left(a_{13}a_{21}\tau_m-a_{11}\tau_q+a_{13}a_{21}\tau_q+\frac{a_{13}a_{22}^2\tau_m}{a_{12}-a_{22}a_{13}}-a_{22}\tau_q\right)\right]t-\frac{a_{13}K_7\tau_m}{\tau_m-\tau_q}\exp\left(-\frac{t}{\tau_m}\right)+K_8\exp\left(-\frac{t}{\tau_q}\right)+(K_1x^2+K_2x)t+K_{11},$$
$$u=K_1\left(a_{21}+\frac{a_{22}^2}{a_{12}-a_{22}a_{13}}\right)t^2+\left[\frac{K_5a_{22}}{a_{12}}-2K_1\left(a_{21}\tau_m+\frac{a_{22}^2\tau_m}{a_{12}-a_{22}a_{13}}\right)\right]t+K_7\exp\left(-\frac{t}{\tau_m}\right)+\left(K_1\frac{a_{22}}{a_{12}-a_{22}a_{13}}x^2+\frac{K_2a_{22}}{a_{12}-a_{22}a_{13}}x\right)t+\left[K_1\left(\frac{a_{12}a_{21}-a_{11}a_{22}}{a_{12}-a_{22}a_{13}}\right)x^4+2K_2\left(\frac{a_{12}a_{21}-a_{11}a_{22}}{a_{12}-a_{22}a_{13}}\right)x^3+6K_5\left(\frac{a_{12}a_{21}-a_{11}a_{22}}{a_{12}}\right)x^2+K_{10}x\right]\Big/12(a_{12}a_{21}-a_{11}a_{22})+K_{12}. \quad (9)$$

When $t=0$, $\theta=-\frac{a_{13}K_7\tau_m}{\tau_m-\tau_q}+K_8+K_{11}$, which is a constant. Therefore, a solution with an even initial temperature field is obtained.

Similarly, assuming: $K_3(a_{11}-a_{21}a_{13})=K_1a_{21}$, $K_4(a_{11}-a_{13}a_{21})=K_2a_{21}$, $K_6a_{11}=K_5a_{21}$, $K_{10}=0$, a solution with an even initial moisture field is obtained, and more details are not given here.

The algebraically explicit analytical solution (8) has terms like $-\frac{a_{13}K_7\tau_m}{\tau_m-\tau_q}\exp\left(-\frac{t}{\tau_m}\right)$ and $K_8\exp\left(-\frac{t}{\tau_q}\right)$ in θ, and $K_7\exp\left(-\frac{t}{\tau_m}\right)$ in u. The absolute values of these terms decrease rapidly with t. Therefore, they are only effective in a time interval not much higher than τ_m or τ_q. It reflects the characteristic of non-Fourier and non-Fick heat and mass transfer process. Since solution (8) includes terms whose denominators are $\tau_m-\tau_q$, it cannot degenerate simply into only Fourier or only Fick case. Because this solution only consists of low-order power functions of time and geometric coordinates, it is especially suitable as a benchmark to check and develop numerical computation. However, the method of separation variables used above cannot lead to the solution with heat wave effect and the preliminary criterion to consider the existence of non-Fourier heat wave effect as in Ref. [13].

2 The second set of analytical solution

As mentioned above, the methods of separation variables can derive different solutions by assuming various specific forms of function constitution. Therefore, we should, according to the characteristic of the equations to be resolved, try various methods of separation variables to derive as many solutions of different forms as possible. For example, assuming $\theta=T_1(t)x+X_1(x)+C_1x^3t$, $u=T_2(t)x+X_2(x)+C_2x^3t$, we can obtain:

$$\begin{aligned}\theta=&3(C_1a_{11}+C_2a_{12}-C_1a_{13}a_{21}-C_2a_{13}a_{22})t^2x\\&+(6C_1a_{13}a_{21}\tau_m+6C_2a_{13}a_{22}\tau_m-6C_1a_{11}\tau_q\\&-6C_2a_{12}\tau_q+6C_2a_{13}a_{22}\tau_q+6C_1a_{13}a_{21}\tau_q\\&+C_3-C_4a_{13})tx-\frac{C_5a_{13}\tau_m}{\tau_m-\tau_q}\exp\left(-\frac{t}{\tau_m}\right)x\\&+C_6\exp\left(-\frac{t}{\tau_q}\right)x+[3(C_2a_{12}-C_2a_{22}a_{13}\\&-C_1a_{22})x^5+10(C_4a_{12}-C_3a_{22})x^3+C_7x]\\&/60(a_{12}a_{21}-a_{11}a_{22})+C_1x^3t+C_8,\\u=&3(C_1a_{21}+C_2a_{22})t^2x+(C_4-6C_1a_{21}\tau_m\\&-6C_2a_{22}\tau_m)tx+C_5\exp\left(-\frac{t}{\tau_m}\right)x\\&+[3(C_1a_{21}+C_2a_{21}a_{13}-C_2a_{11})x^5\\&+10(C_3a_{21}-C_4a_{11})x^3+C_9x]\\&/60(a_{12}a_{21}-a_{11}a_{22})+C_2x^3t+C_{10},\end{aligned}\tag{10}$$

where C_1—C_{10} are arbitrary constants.

Similar to the above section, assuming: $C_2(a_{12}-a_{22}a_{13})=C_1a_{22}$, $C_4a_{12}=C_3a_{22}$, $C_6=\frac{C_5a_{13}\tau_m}{\tau_m-\tau_q}$ and $C_7=0$, solution (10) can be simplified as:

$$\begin{aligned}\theta=&3C_1(a_{11}+a_{22}-a_{13}a_{21})t^2x+\Bigg[6C_1\Bigg(a_{13}a_{21}\tau_m\\&-a_{11}\tau_q+a_{13}a_{21}\tau_q+\frac{a_{13}a_{22}^2\tau_m}{a_{12}-a_{22}a_{13}}-a_{22}\tau_q\Bigg)\\&+C_3\left(1-\frac{a_{13}a_{22}}{a_{12}}\right)\Bigg]tx\\&+\frac{C_5a_{13}\tau_m}{\tau_m-\tau_q}\left[\exp\left(-\frac{t}{\tau_q}\right)-\exp\left(-\frac{t}{\tau_m}\right)\right]x\\&+C_1x^3t+C_8,\\u=&3C_1\left(a_{21}+\frac{a_{22}^2}{a_{12}-a_{22}a_{13}}\right)t^2x\\&+\Bigg[\frac{C_3a_{22}}{a_{12}}+\frac{C_2a_{22}}{a_{12}-a_{13}a_{22}}\\&-6C_1\left(a_{21}\tau_m+\frac{a_{22}^2\tau_m}{a_{12}-a_{22}a_{13}}\right)\Bigg]tx\\&+C_5\exp\left(-\frac{t}{\tau_m}\right)x+\Bigg[3C_1\left(\frac{a_{12}a_{21}-a_{11}a_{22}}{a_{12}-a_{22}a_{13}}\right)x^5\\&+10C_3\left(\frac{a_{12}a_{21}-a_{11}a_{22}}{a_{12}}\right)x^3+C_9x\Bigg]\\&/60(a_{12}a_{21}-a_{11}a_{22})+\frac{C_1a_{22}}{a_{12}-a_{22}a_{13}}x^3t+C_{10}.\end{aligned}\tag{11}$$

For solution (11), when $t=0$, $\theta=C_8$, which is a constant; when $x=0$, $\theta=C_8$, $u=C_{10}$. It means that the temperature field is initially even and the temperature and moisture are constants at $x=0$. Therefore, similar to the solution in the previous paragraph, the boundary condition and the initial condition of the solution are simple and suitable to be a benchmark solution. The physics experiments would also be easy to carry out.

3 Conclusions

Based on the method of separation variables with addition developed in recent years, new methods of separation variables are proposed:

$$f=T_1(t)+X_1(x)+(K_1x^2+K_2x)t,$$
$$f=T_1(t)x+X_1(x)+C_1x^3t.$$

Apply them to the non-Fourier and non-Fick heat and mass transfer equation set of porous media, two sets of algebraically explicit analytical solutions have been obtained. The boundary condition and the initial condition of these solutions can be constant. The forms of the solutions are also relatively simple. Besides their irreplaceable theoretical meaning for the study of heat and mass transfer in the rapid drying process of porous media, these analytical solutions can also be applied as benchmark solutions to check the numerical computation solutions.

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